Section 16.4:
Green's Theorem

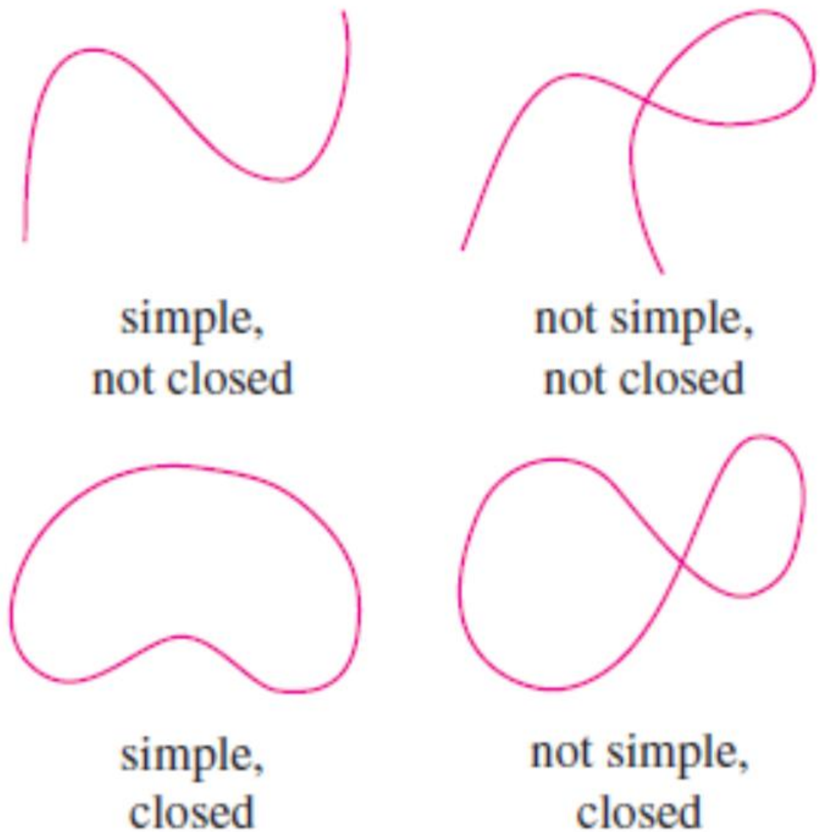
What We'll Learn In Section 16.4

1. Green's Theorem
2. Green's Theorem in Reverse
3. Extended Version of Green's Theorem

1. Green's Theorem

<u>Def</u>: Let C be a curve in $\mathbb{R}^2$.

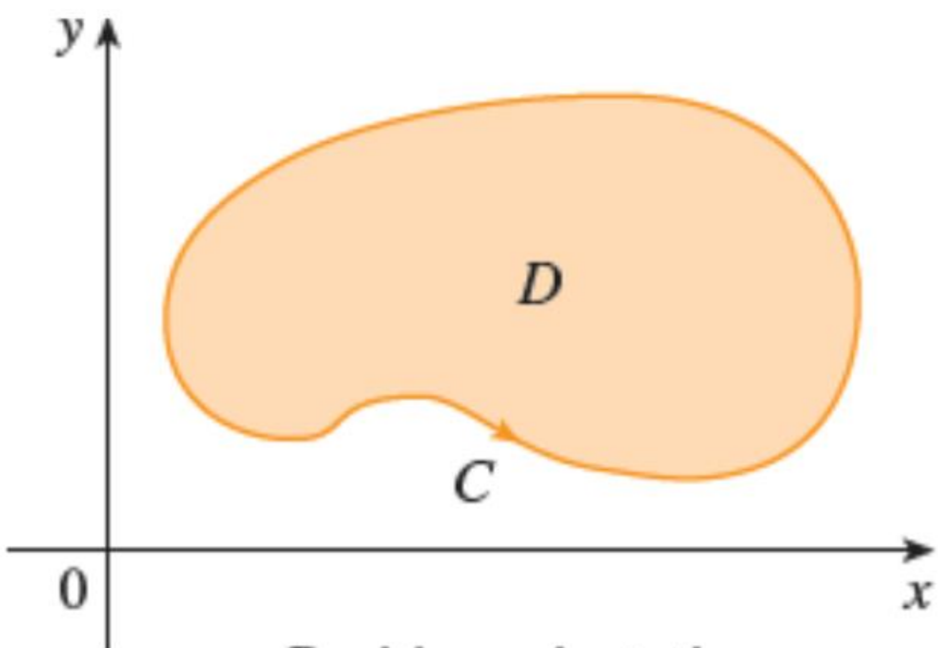
1. C is <u>simple</u> if it doesn't cross itself (except possibly at the endpoints of the curve)
2. C is <u>closed</u> if the endpoints of the curve are the same.



1. Green's Theorem

Def: Let C be a curve in $\mathbb{R}^2$.

3. A simple closed curve C is positively oriented if it is oriented in the counterclockwise direction.



Positive orientation

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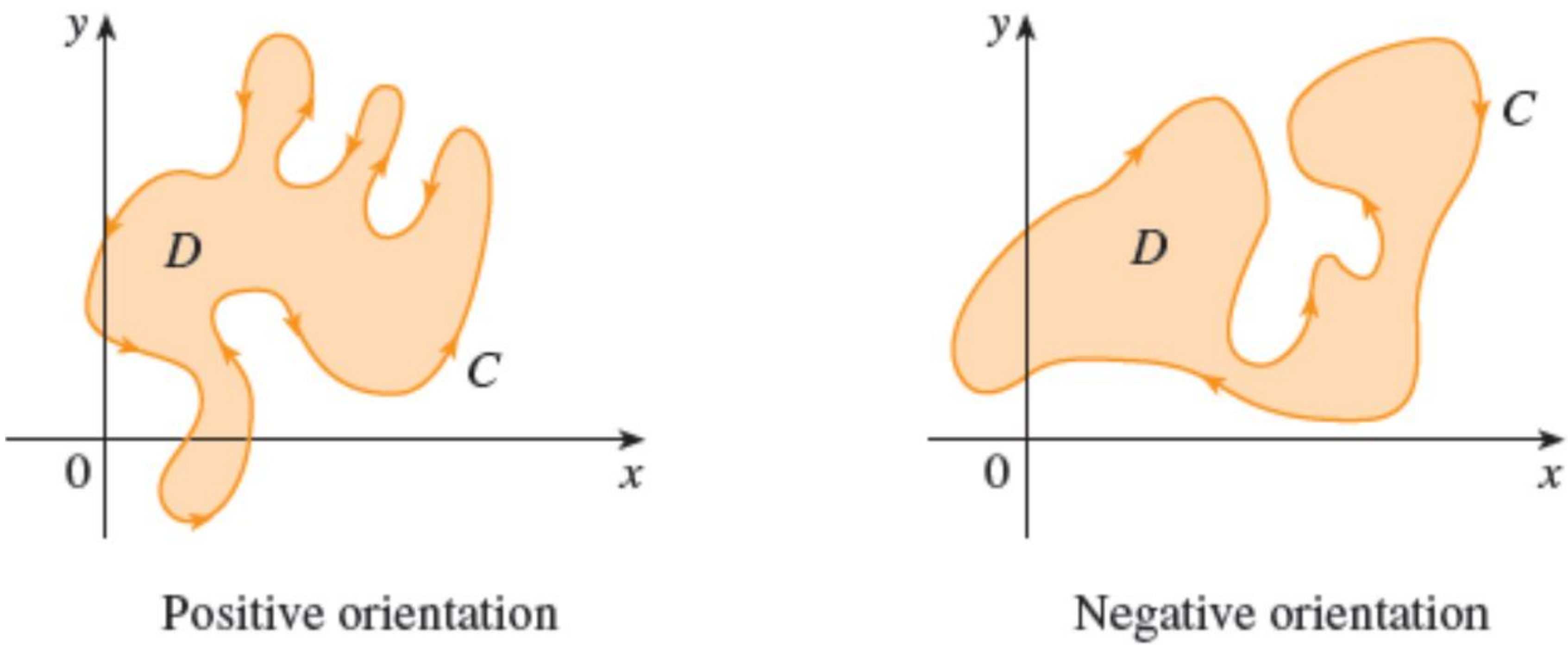

1. Green's Theorem

Green's Theorem

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

1. Green's Theorem

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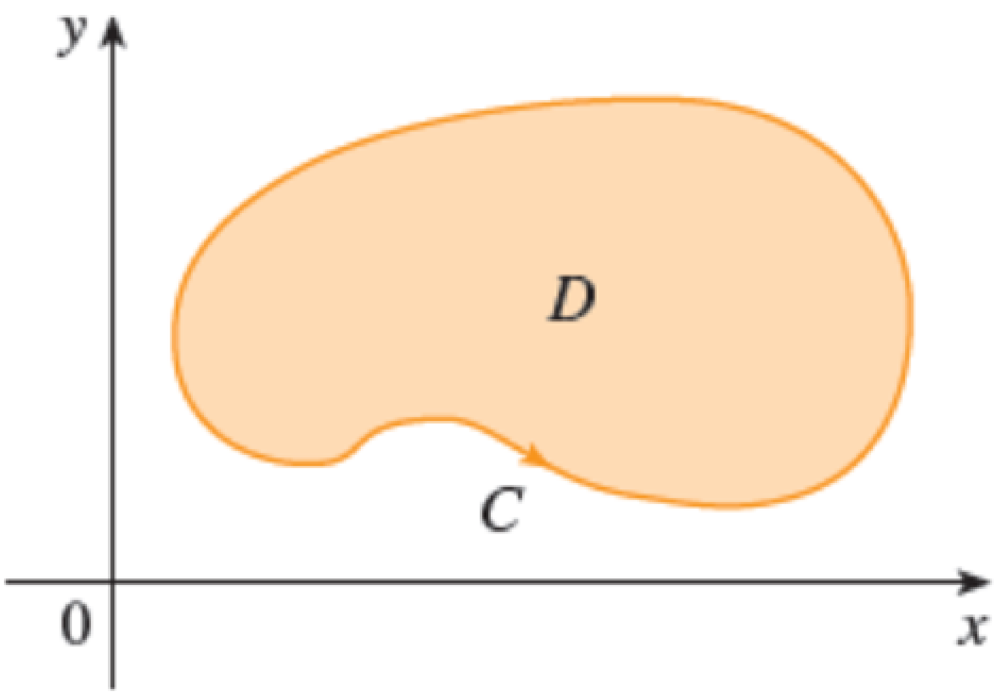

1. Green's Theorem

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

<u>Notes:</u>

1) The left side (above) is the same as $\int \vec{F} \cdot d\vec{r}$ where $\vec{F} =< P(x,y),\ Q(x,y) >$

2) The notation $\oint Pdx + Qdy$ means you are integrating along a simple closed curve that is positively oriented

1. Green's Theorem

$$\int_C P\,dx + Q\,dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

<u>Notes:</u>

3) ∂D stands for the boundary of region D (i.e. the curve C) and so Green's Theorem can be written as…

$$\iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy$$

1. Green's Theorem

Ex 1: Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from (0,0) to (1,0), from (1,0) to (0,1), and from (0,1) to (0,0).

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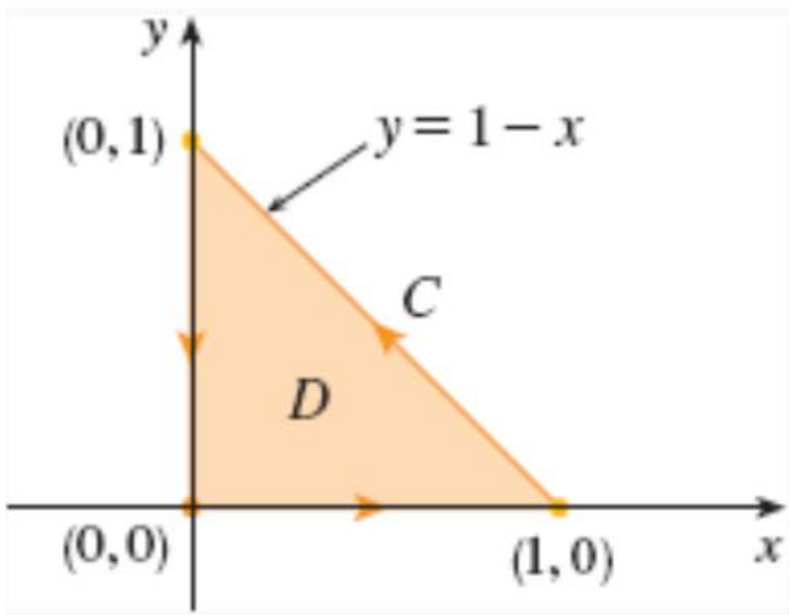


1. Green's Theorem

<u>Ex 2</u>: Evaluate $\oint_C \left(3y - e^{\sin x}\right) dx + \left(7x + \sqrt{y^4 + 1}\right) dy$, where C is the circle $x^2 + y^2 = 9$.

2. Green's Theorem In Reverse

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$$

- For instance, if it is known that $P(x,y) = Q(x,y) = 0$ on the curve C, then Green's Theorem gives

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy = 0$$

2. Green's Theorem In Reverse

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P\,dx + Q\,dy$$

- Using the fact that $\iint_D 1 \; dA = A(D)$, we can use Green's Theorem to compute areas by doing line integrals!
- There are many choices for P and Q...

2. Green's Theorem In Reverse

$$\iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \; = \; \int_C P\,dx + Q\,dy$$

$$P(x,y) = 0 \qquad P(x,y) = -y \qquad P(x,y) = -0.5y$$

$$Q(x,y) = x \qquad Q(x,y) = 0 \qquad Q(x,y) = 0.5x$$

This gives many different formulas for the area of *D* ...

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$$

2. Green's Theorem In Reverse

Ex 3: Use Green's Theorem (in reverse) to find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

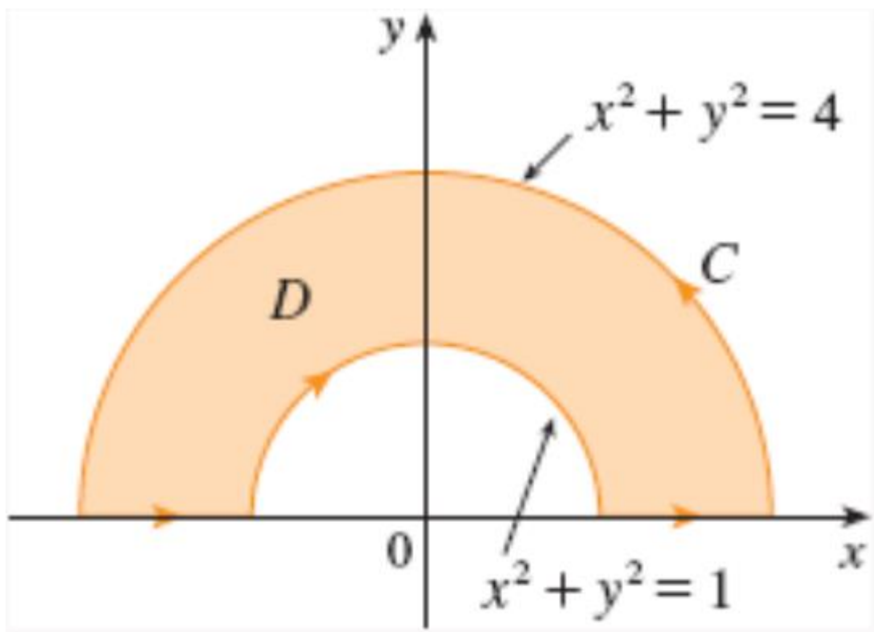
Planimeter?

3. Extended Version of Green's Theorem

Ex 4: Evaluate $\oint_C y^2\,dx + 3xy\;dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

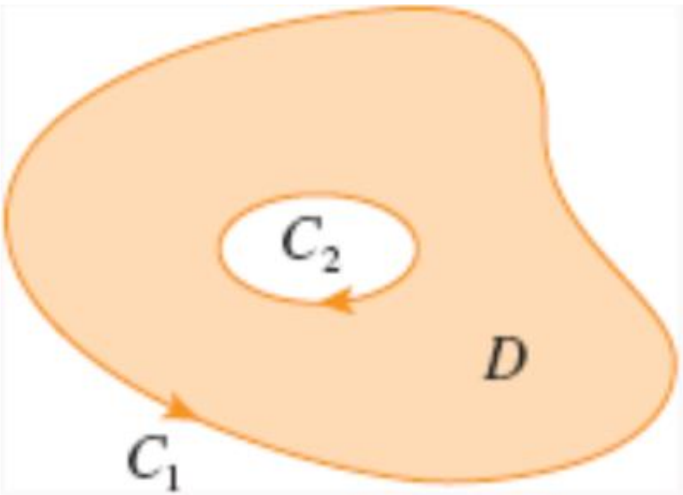
3. Extended Version of Green's Theorem

Ex 4: Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2+y^2=1$ and $x^2+y^2=4$.



3. Extended Version of Green's Theorem

Green's Theorem can be extended to regions that are not simply connected (regions with holes in them).



How?
First, positively orient the hole(s).
Then…

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy = \int_C P\,dx + Q\,dy$$

3. Extended Version of Green’s Theorem

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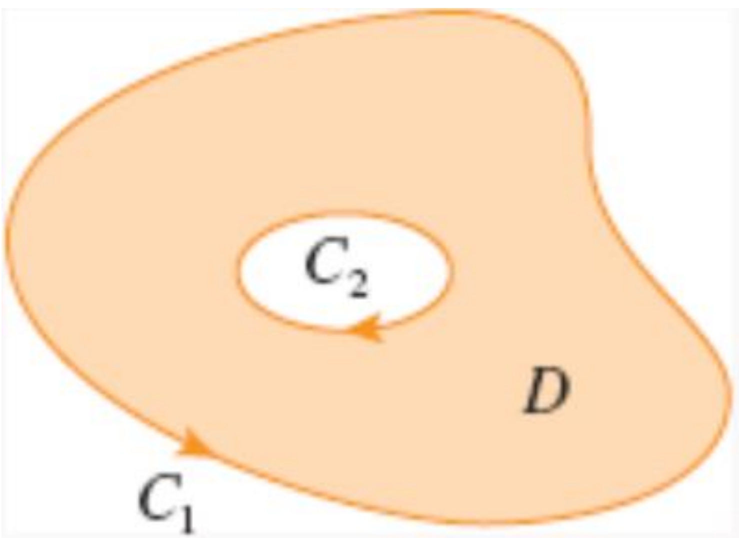

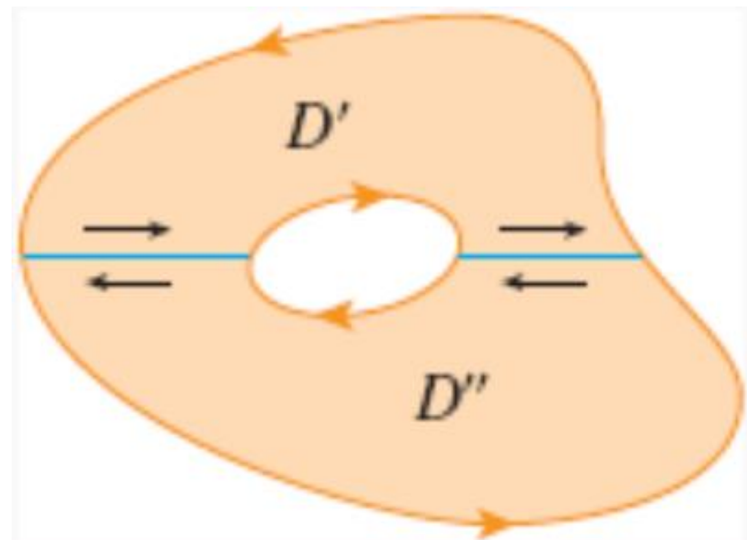

3. Extended Version of Green's Theorem

Ex 5: If $\vec{F} = < -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} >$, show that $\int_C \vec{F} \cdot d\vec{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.